North Sydney Boys High School

MATHEMATICS (EXTENSION 1)

2011 HSC Course Assessment Task 1

November 29, 2010

General instructions

- Working time – 50 minutes.
- Commence each new question on a new page. Write on both sides of the paper.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

Class (please ✔)

- ◯ 12M3A – Mr Lam
- ◯ 12M3B – Mr Weiss
- ◯ 12M3C – Mr Trenwith
- ◯ 12M4A – Mr Fletcher
- ◯ 12M4B – Mr Ireland
- ◯ 12M4C – Mr Rezcallah

NAME: .... **# PAGES USED**: ....

Marker's use only.

QUESTION	1	2	3	4	5	6	**Total**	%
MARKS	$\overline{9}$	$\overline{9}$	$\overline{9}$	$\overline{7}$	$\overline{10}$	$\overline{9}$	$\overline{53}$	

Question 1 (9 Marks) Commence a NEW page. **Marks**

(a) Given that $\sin A = \frac{2}{3}$ and A is acute, find the value of $\sin 2A$. **2**

(b) Find the exact value of $\cos 105°$. **2**

(c) Show that $\dfrac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x$. **2**

(d) Solve the equation $2\sin^2 x = 1 + \cos 2x$ for $0 \leq x \leq 2\pi$. **3**

Question 2 (9 Marks) Commence a NEW page. **Marks**

(a) For what value of k does the equation **3**

$$x^2 + (k - 4)x + 9 = 0$$

have real roots?

(b) Solve the equation $x^4 = 4(x^2 + 3)$. **3**

(c) Write $2x^2 - 3x + 3$ in the form $A(x - 1)^2 + B(x - 1) + C$. **3**

Question 3 (9 Marks) Commence a NEW page. **Marks**

(a) A monic polynomial of degree 3 has roots -1, 1 and 2. Write the equation of the polynomial in the form **2**

$$P(x) = ax^3 + bx^2 + cx + d$$

(b) The polynomial $P(x) = px^3 + 5x^2 - 3p$ has $(x - 2)$ as a factor. Find the value of p. **2**

(c) Let $f(x) = x^3 + 2x^2 + 5x - 4$.

i. Show that $f(x)$ has a root between $x = 0$ and $x = 1$. **2**

ii. Taking $x = 0.5$ as an approximation to this root, use one application of Newton's method to find a better approximation to the root. **3**

Question 4 (7 Marks) Commence a NEW page. **Marks**

(a) Find the Cartesian equation of the point $P(t+1, 2t^2+1)$. **2**

(b) The curves $y=(x-1)^2$ and $y=(x+1)^2$ intersect at the point Q.

i. Find the coordinates of Q. **2**

ii. Find the acute angle between the tangents to the curves at Q, giving your answer to the nearest degree. **3**

Question 5 (10 Marks) Commence a NEW page. **Marks**

(a) When the polynomial $P(x)$ is divided by (x^2-1), the remainder is $x-4$. Find the remainder when $P(x)$ is divided by $(x+1)$. **2**

(b) i. Find the equation of the locus of the point $P(x,y)$ which moves such that its distance from $A(-3,2)$ is twice its distance from $B(3,-4)$. **3**

ii. Describe this locus geometrically. **2**

(c) One of the roots of $2x^3+x^2-15x-18=0$ is positive and equal to the product of the other two roots. **3**

Find all the roots of this equation.

Question 6 (9 Marks) Commence a NEW page. **Marks**

$P(2p, p^2)$ and $Q(2q, q^2)$ lie on the parabola $x^2=4y$.

(a) Derive the equation of the normal to the parabola at P. **2**

(b) The chord PQ passes through the point $(0,-2)$. Find the equation of the chord PQ and show that $pq=2$. **3**

(c) The normals at P and Q intersect at R. Show that R lies on the parabola. **4**

End of paper.

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Suggested Solutions

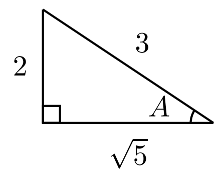
Question 1 (Fletcher)

(a) (2 marks)

✓ [1] for triangle (or equivalent).

✓ [1] for final answer in exact form.

$$\sin A = \frac{2}{3}$$



$$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ &= 2\times\frac{2}{3}\times\frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}\end{aligned}$$

(b) (2 marks)

✓ [1] for correct substitution.

✓ [1] for final answer.

$$\begin{aligned}\cos 105^\circ &= \cos(60^\circ + 45^\circ)\\ &= \cos 60^\circ\cos 45^\circ - \sin 60^\circ \sin 45^\circ\\ &= \left(\frac{1}{2}\times\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}\right)\\ &= \frac{1-\sqrt{3}}{2\sqrt{2}}\ \left(= \frac{\sqrt{2}-\sqrt{6}}{4}\right)\end{aligned}$$

(c) (2 marks)

✓ [1] for $\frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$.

✓ [1] for final answer.

$$\begin{aligned}\frac{1-\tan^2 x}{1+\tan^2 x} &= \frac{1-\frac{\sin^2 x}{\cos^2 x}}{1+\frac{\sin^2 x}{\cos^2 x}}\,\frac{\times\cos^2 x}{\times \cos^2 x}\\ &= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}\\ &= \cos 2x\end{aligned}$$

(d) (3 marks)

✓ [1] for $4\sin^2 x = 1$, or equivalent.

✓ [1] for $\sin x = \pm\frac{1}{\sqrt{2}}$.

✓ [1] for all four solutions. If only $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ is written, a maximum of (2 marks) are awarded.

$$\begin{aligned}2\sin^2 x &= 1 + \cos 2x\\ 2\sin^2 x &= 1 + \left(1 - 2\sin^2 x\right)\\ 4\sin^2 x &= 1\\ \sin^2 x &= \frac{1}{2} \quad\Rightarrow\quad \sin x = \pm\frac{1}{\sqrt{2}}\\ \therefore x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\end{aligned}$$

Question 2 (Rezcallah)

(a) (3 marks)

✓ [1] for $(k-4)^2 - 36 \geq 0$.

✓ [1] for $(k-10)(k+2) \geq 0$.

✓ [1] for final answer.

$$x^2 + (k-4)x + 9 = 0$$

$$\Delta = (k-4)^2 - 4(9) = (k-4)^2 - 36$$

There will be real roots when $\Delta \geq 0$:

$$\begin{aligned}(k-4)^2 - 36 &\geq 0\\ (k-4-6)(k-4+6) &\geq 0\\ (k-10)(k+2) &\geq 0\\ \therefore k \geq 10 &\text{ or } k \leq -2\end{aligned}$$

(b) (3 marks)

✓ [1] for making the substitution and transforming into quadratic (or equivalent).

✓ [1] for $x^2 = -2, 6$ (or equivalent).

✓ [1] for final answer.

$$\begin{aligned}x^4 &= 4(x^2+3)\\ x^4 - 4x^2 - 12 &= 0\end{aligned}$$

Let $m = x^2$,

$$m^2 - 4m - 12 = 0$$
$$(m-6)(m+2) = 0$$
$$\therefore m = -2, 6$$
$$\therefore x^2 = -2, 6$$

$x^2 = -2$ has no real solutions.

$$\therefore x = \pm\sqrt{6}$$

(c) (3 marks)

✓ [1] for each correct value of A, B and C and writing it in the form specified.

$$2x^2 - 3x + 3 \equiv A(x-1)^2 + B(x-1) + C$$

By inspection, $A = 2$. Let $x = 1$,

$$2 - 3 + 3 = C$$
$$\therefore C = 2$$

Letting $x = 0$,

$$3 = 2(-1)^2 + B(-1) + 2$$
$$3 = 2 - B + 2$$
$$-B = -1$$
$$\therefore B = 1$$
$$\therefore 2x^2 - 3x + 3 \equiv 2(x-1)^2 + 1(x-1) + 2$$

Question 3 (Lam)

(a) (2 marks)

✓ [-1] for each error.

$$P(x) = ax^3 + bx^2 + cx + d$$

- As $P(x)$ is monic, $\therefore a = 1$.
- The roots are $\alpha = -1$, $\beta = 1$ and $\gamma = 2$.
- Apply the sum of roots,

$$\alpha + \beta + \gamma = -\frac{b}{a}$$
$$-1 + 1 + 2 = -b$$
$$\therefore b = 2$$

Apply the sum of pairs of roots,

$$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$$
$$(-1)(1) + \cancel{(-1)(2)} + \cancel{1(2)} = c$$
$$\therefore c = -1$$

Apply the product of roots,

$$\alpha\beta\gamma = -\frac{d}{a}$$
$$(-1)(1)(2) = -d$$
$$\therefore d = 2$$
$$\therefore P(x) = x^3 - 2x^2 - x + 2$$

Alternatively, expand $(x^2-1)(x-2)$.

(b) (2 marks)

✓ [1] for applying the factor theorem and evaluating $P(2)$.

✓ [1] for final answer.

$$P(x) = px^3 + 5x^2 - 3p$$

By the factor theorem,

$$P(2) = 0$$
$$\therefore 2^3p + 5(2^2) - 3p = 0$$
$$8p + 20 - 3p = 0$$
$$5p + 20 = 0$$
$$\therefore p = -4$$

(c) i. (2 marks)

✓ [1] for evaluating $f(0)$ and $f(1)$.

✓ [1] for final statement (or equivalent).

$$f(x) = x^3 + 2x^2 + 5x - 4$$
$$f(0) = -4$$
$$f(1) = 1 + 2 + 5 - 4$$
$$= 4$$

As $f(0) < 0$ and $f(1) > 0$ and f is continuous for all x, therefore $f(x)$ has a root between $x = 0$ and $x = 1$.

ii. (3 marks)

✓ [1] for correct differentiation.

✓ [2] for final answer. Deduct [1] for each error.

$$f(x) = x^3 + 2x^2 + 5x - 4$$
$$f'(x) = 3x^2 + 4x + 5$$
$$f\left(\frac{1}{2}\right) = \frac{1}{8} + 2\left(\frac{1}{4}\right) + \frac{5}{2} - 4 = -\frac{7}{8}$$
$$f'\left(\frac{1}{2}\right) = 3\left(\frac{1}{4}\right) + 4\left(\frac{1}{2}\right) + 5 = \frac{31}{4}$$
$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= \frac{1}{2} - \frac{-\frac{7}{8}}{\frac{31}{4}} \\ &= \frac{19}{31} \approx 0.6129\cdots \end{aligned}$$

Note: $f\left(\frac{19}{31}\right) \approx 0.04\cdots$.
Hence $x = \frac{19}{31} \approx 0.6129$ is a better approximation of the root.

Question 4 (Ireland)

(a) (2 marks)

✓ [1] for substituting $t = x - 1$ into $y = 2t^2 + 1$.

✓ [1] for final answer $y = 2(x-1)^2 + 1$.

$$x = t + 1 \qquad y = 2t^2 + 1$$

Rearrange $x = t + 1$ and substitute into $y = 2t^2 + 1$:

$$\begin{aligned} \therefore y &= 2(x-1)^2 + 1 \\ &= 2(x^2 - 2x + 1) + 1 \\ &= 2x^2 - 4x + 3 \end{aligned}$$

(b) i. (2 marks)

✓ [1] for each value of x and y.

Equate to solve simultaneously:

$$\begin{cases} y = (x-1)^2 \\ y = (x+1)^2 \end{cases}$$
$$(x+1)^2 - (x-1)^2 = 0$$
$$\Big[(x+1) - (x-1)\Big]\Big[(x+1) + (x-1)\Big] = 0$$
$$(2)(2x) = 0$$
$$\therefore x = 0$$
$$\therefore y = (0-1)^2 = 1$$
$$Q(0, 1)$$

ii. (3 marks)

✓ [1] for both values of m_1 and m_2.

✓ [1] for correct substitution into $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$. Max [1] for error in this formula.

✓ [1] for final answer.

$$y = (x-1)^2 \qquad \frac{dy}{dx} = 2(x-1)$$
$$y = (x+1)^2 \qquad \frac{dy}{dx} = 2(x+1)$$

At $x = 0$, the gradients of the tangents to the curves are

$$m_1 = 2(-1) = -2 \qquad m_2 = 2(1) = 2$$

Applying the angle between two lines formula,

$$\begin{aligned} \tan\theta &= \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \\ &= \left|\frac{2 - (-2)}{1 + (2)(-2)}\right| = \left|\frac{4}{-3}\right| \\ \therefore \theta &= 53^\circ \text{ (nearest degree)} \end{aligned}$$

Question 5 (Trenwith)

(a) (2 marks)

✓ [1] for rewriting $P(x)$ as the division identity.

✓ [1] for final answer.

$$\begin{aligned} P(x) &= (x^2-1)Q(x) + (x-4) \\ &= (x-1)(x+1)Q(x) + (x-4) \end{aligned}$$

Apply the remainder theorem for divison by $x+1$:

$$P(-1) = 0 + (-1-4) = -5$$

Hence the remainder when dividing $P(x)$ by $(x+1)$ is $R=-5$.

(b) i. (3 marks)

✓ [1] for $PA^2 = 4PB^2$.

✓ [1] for substituting expression into $PA^2 = 4PB^2$.

✓ [1] for $3x^2 - 30x + 27 + 3y^2 + 36y + 60 = 0$.

$$\begin{aligned} PA &= \sqrt{(x+3)^2 + (y-2)^2} \\ PB &= \sqrt{(x-3)^2 + (y+4)^2} \\ PA &= 2PB \\ \therefore PA^2 &= 4PB^2 \\ \therefore (x+3)^2 + (y-2)^2 & \\ &= 4(x-3)^2 + 4(y+4)^2 \\ 4(x-3)^2 + 4(y+4)^2 & \\ &- (x+3)^2 - (y-2)^2 = 0 \\ 4\left(x^2-6x+9\right) + 4\left(y^2+8y+16\right) & \\ - \left(x^2+6x+9\right) - \left(y^2-4y+4\right) &= 0 \\ 3x^2 - 30x + 27 + 3y^2 + 36y + 60 &= 0 \\ 3x^2 - 30x + 3y^2 + 36y + 87 &= 0 \end{aligned}$$

ii. (2 marks)

✓ [1] for circle.

✓ [1] for correct centre and radius.

$$\begin{aligned} \underbrace{3x^2 - 30x + 27 + 3y^2 + 36y + 60}_{\div 3} &= \underset{\div 3}{0} \\ x^2 - 10x + \underset{-9}{9} + y^2 + 12y + \underset{-20}{20} &= \underset{-29}{0} \\ x^2 - 10x + 25 + y^2 + 12y + 36 & \\ &= -29 + 25 + 36 \\ (x-5)^2 + (y+6)^2 &= 32 \end{aligned}$$

Circle with centre $(5,-6)$ and radius $r = 4\sqrt{2}$.

(c) (3 marks)

✓ [1] for correctly finding the product of roots.

✓ [1] for $2\alpha^2 + 7\alpha + 6 = 0$.

✓ [1] for final answers.

$$2x^3 + x^2 - 15x - 18 = 0$$

Let the roots be α, β and $\alpha\beta$, where $\alpha\beta > 0$. Apply the product of roots,

$$\alpha\beta \times \alpha\beta = \alpha^2\beta^2 = -\frac{d}{a} = 9$$

$$\therefore \alpha\beta = 3 \qquad (5.1)$$

Hence one of the roots is 3. Rearrange,

$$\therefore \alpha = \frac{3}{\beta} \qquad (5.2)$$

Apply the sum of roots,

$$\alpha + \beta + \alpha\beta = -\frac{b}{a} = -\frac{1}{2} \qquad (5.3)$$

Substitute (5.2) into (5.3),

$$\begin{aligned} \underbrace{\alpha + \frac{3}{\alpha} + 3}_{\times 2\alpha} &= \underset{\times 2\alpha}{-\frac{1}{2}} \\ 2\alpha^2 + 6 + 6\alpha &= -\alpha \\ 2\alpha^2 + 7\alpha + 6 &= 0 \\ (2\alpha+3)(\alpha+2) &= 0 \\ \therefore \alpha &= -\frac{3}{2}, -2 \end{aligned}$$

Hence the roots are $-\frac{3}{2}$, -2 and 3.

Question 6 (Weiss)

(a) (2 marks)

- ✓ [1] for derivation of gradient of normal.
- ✓ [1] for final answer, or equivalent in gradient-intercept form.

$$P(2p, p^2) \qquad Q(2q, q^2)$$

Rearrange $x^2 = 4y$,

$$y = \frac{1}{4}x^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}x$$

At $x = 2p$,

$$\frac{dy}{dx} = \frac{1}{2} \times 2p = p$$
$$\therefore m_\perp = -\frac{1}{p}$$

Applying the point gradient formula,

$$\frac{y - p^2}{x - 2p} = -\frac{1}{p}$$
$$py - p^3 = -x + 2p$$
$$\therefore x + py = p^3 + 2p$$

(b) (3 marks)

- ✓ [1] for correct application of two-point formula.
- ✓ [1] for correct equation of PQ.
- ✓ [1] for correct substitution of $(0, -2)$ into equation of PQ and conclusion.

Apply the two point formula to find the equation of chord PQ:

$$\frac{y - q^2}{x - 2q} = \frac{p^2 - q^2}{2p - 2q} = \frac{\cancel{(p-q)}(p+q)}{2\cancel{(p-q)}}$$
$$= \frac{p+q}{2}$$
$$y - q^2 = \left(\frac{p+q}{2}\right)x - 2q\left(\frac{p+q}{2}\right)$$
$$y - \cancel{q^2} = \left(\frac{p+q}{2}\right)x - pq - \cancel{q^2}$$
$$\therefore y = \left(\frac{p+q}{2}\right)x - pq$$

As the chord PQ passes through $(0, -2)$,

$$-2 = 0 - pq$$
$$\therefore pq = 2$$

(c) (4 marks)

- ✓ [1] for $y = p^2 + q^2 + pq + 2$.
- ✓ [1] for $x = -2(p+q)$.
- ✓ [1] for use of $pq = 2$ in y coordinate.
- ✓ [1] for correct substitution and conclusion.

From part (a), the normals at P and Q are

$$\begin{cases} x = -py + p^3 + 2p \\ x = -qy + q^3 + 2q \end{cases}$$

Solving simultaneously by equating,

$$-py + p^3 + 2p = -qy + q^3 + 2q$$
$$py - qy = p^3 - q^3 + 2p - 2q$$
$$y(p-q) = \left(p^3 - q^3\right) + 2(p-q)$$
$$y\cancel{(p-q)} = \cancel{(p-q)}(p^2 + \cancel{pq}^{\,2} + q^2) + 2\cancel{(p-q)}$$
$$\therefore y = p^2 + q^2 + 2 + 2$$
$$= p^2 + q^2 + 4$$
$$= p^2 + 2pq + q^2$$
$$= (p+q)^2$$

Substitute to find x,

$$x + p(p+q)^2 = p^3 + 2p$$
$$x = -p(p+q)^2 + p^3 + 2p$$
$$= -p(p^2 + 2pq + q^2) + p^3 + 2p$$
$$= \cancel{-p^3} - 2p^2q - pq^2 + \cancel{p^3} + 2p$$
$$= -pq(2p + q) + 2p$$
$$= -2(2p + q) + 2p$$
$$= -4p - 2q + 2p$$
$$= -2p - 2q = -2(p+q)$$

Check that R with coordinates $x = -2(p+q)$ and $y = (p+q)^2$ lies on the parabola $x^2 = 4y$.

$$x = -2(p+q)$$
$$\therefore p + q = -\frac{1}{2}x$$

Substitute to y:

$$y = (p+q)^2$$
$$= \left(-\frac{1}{2}x\right)^2 = \frac{1}{4}x^2$$

$\therefore R$ lies on the parabola as well.